Gravitational Wave Chirp Search: Economization of PN Matched Filter Bank via Cardinal Interpolation

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The final inspiral phase in the evolution of a compact binary consisting of black holes and/or neutron stars is among the most probable events that a network of ground-based interferometric gravitational wave detectors is likely to observe. Gravitational radiation emitted during this phase will have to be dug out of noise by matched-filtering (correlating) the detector output with a bank of several 10^5 templates, making the computational resources required quite demanding, though not formidable. We propose an interpolation method for evaluating the correlation between template waveforms and the detector output and show that the method is effective in substantially reducing the number of templates required. Indeed, the number of templates needed could be a factor ~ 4 smaller than required by the usual approach, when the minimal overlap between the template bank and an arbitrary signal (the so-called *minimal match*) is 0.97. The method is amenable to easy implementation, and the various detector projects might benefit by adopting it to reduce the computational costs of inspiraling neutron star and black hole binary search.

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I. INTRODUCTION

Black hole (BH) and neutron star (NS) binaries end their lives by emitting a chirp-like burst of gravitational wave (GW) radiation. The inspiral radiation from NS-NS sources located within a distance of 20 Mpc, and BH-BH sources within 200 Mpc, should be observable by a first-generation network of ground-based long-baseline interferometric GW detectors [1]. Correlating the detector output (data) with a family of expected waveforms (templates) and using the largest correlator as a detection statistic, provides the best detection strategy for Gaussian colored stationary noise (maximum likelihood, matched filtering [2]), but as we will not know the waveform parameters beforehand, a bank of 10-100 thousands of templates will have to be used so as not to miss out any source [3]. Data analysis groups have been, therefore, steadily looking for data analysis economization strategies.

In setting up a bank of templates to search for a signal with uknown parameters the quantity of interest [4] is the (frequency domain) scalar product between the signal $h(\cdot)$ and the template(s) $g(\cdot)$. The scalar product defined in this context is similar to the usual vector space definition, except that the measure is derived from the detector noise power spectral density $S_n(f)$:

$$\langle h, g\rangle \equiv 2\left[\int_0^\infty \tilde{h}(f)\tilde{g}^*(f)\frac{df}{S_n(f)} + \text{C.C.}\right], \tag{1.1}$$

where C.C. denotes the complex conjugate. It is customary to use unit-norm templates, for which $||g|| \equiv \langle g, g\rangle^{1/2} = 1$. Under this assumption the scalar product is also called *deflection*, and denoted by d [5]. The normalized deflection obtained by further dividing $\langle g, g\rangle$ by $||h||$ is the *overlap* or ambiguity function. It is a real number in $[0, 1]$, and is a measure of how similar the template and the signal are.

Given a binary inspiral signal and a template, the (nominal) times at which they coalesce and the phases at the times of coalescence are unimportant for the purpose of data analysis. We can, therefore, preliminarily maximise the overlap over these parameters [6]. Such a maximised overlap is called the *match* [7], and is a function of the remaining (intrinsic) source and template parameters [8].

The minimal match Γ is the smallest *overlap* of a signal of arbitrary parameters with the template closest to it in the template bank [9]. It is immediately related to the fraction $(1-\Gamma^3)$ of potentially observable sources which *might* be missed out [3].

In a recent paper [10] it has been pointed out that (under certain assumptions) the match is a quasi band-limited function of the difference between the intrinsic source and template parameters. As such, it can be *approximated* by a cardinal imterpolating expansion, which uses only *a fraction* of the templates needed by the std. lattice for a

prescribed minimal match. The same *economized* template set can be used to build a cardinal-interpolated formula for the partially maximised correlator, under the same minimal match constraint.

This was shown in [10] for the simplest (but rather unrealistic) case of Newtonian (0PN) chirp signals. At the lowest Newtonian (0PN) order an inspiral waveform depends only on a single parameter [11]. In this one-parameter template space, the cardinal-interpolation method [10] leads to a reduction in the number of templates by a factor ≈ 1.4 for $\Gamma = 0.97$, and the resulting template density is close to an absolute minimum set by the theory of quasi-band-limited functions. As anticipated in [10], one expects an even *larger* reduction in the number of templates for multi-parameter PN models, in view of well known properties of cardinal interpolating expansion in several dimensions (see references quoted in [10]).

In this Rapid Communication we present results for the case of first-post-Newtonian (1PN) signals and templates, which support the above expectation. We show that in the two-dimensional 1PN template space the reduction in the number of templates is by a factor ≈ 4 for $\Gamma = 0.97$. A similar reduction in the computational resources is implied, which we believe the interferometric detector projects such as TAMA, GEO, LIGO and VIRGO might benefit from. In the following we present only essential ideas and results, deferring the details to a longer version. Throughout this paper we use geometrized units (i.e., $G = c = 1$).

II. THE POST-NEWTONIAN WAVEFORM

We shall use the *restricted* PN approximation, where the amplitude of the waveform is kept to lowest (Newtonian) PN order, while the phase is expanded to the highest PN order available [12].

The evolution of the instantaneous orbital phase φ of an (adiabatically) inspiraling compact circular-orbit binary is obtained (in parametric form) by solving a pair of ordinary differential equations [13]:

$$\frac{dt}{dv} = -\frac{m}{F}\frac{dE}{dv}, \qquad \frac{d\varphi}{dv} = -\frac{v^3}{F}\frac{dE}{dv}, \tag{2.1}$$

where v is the gauge independent relative velocity of the two stars, $F(v)$ is the gravitational wave flux (luminosity), and $E(v)$ the dimensionless relativistic binding energy of the system, related by $F(v) = -m(dE/dt)$. The instantaneous gravitational wave frequency f_{GW} is related to the orbital phase, and hence via (2.1) to v, by:

$$f_{GW} = \pi^{-1}\frac{d\varphi}{dt} = \frac{v^3}{\pi m}. \tag{2.2}$$

The flux and energy functions are presently known to order v^5 (2.5 PN) for non-spinning binaries [14].

The gravitational wave (dominant, $m = 2$ multipole, $2nd$ harmonic of orbital frequency) emitted by an inspiraling compact binary and sensed by an interferometric antenna is described by the waveform [15]:

$$h(t) = 4C\eta\frac{m}{r}v^2(t)\cos[2\varphi(t)], \tag{2.3}$$

where r is the distance to the source, and C is a constant in the range $[0, 1]$ dependent on the relative source/detector orientation with r.m.s value $2/5$ (average over all orientations and wave polarisations) [16].

It is straightforward to obtain an explicit frequency-domain (Fourier transform) representation of the waveform (2.3) using the stationary phase approximation [17, 18]:

$$\tilde{h}(f) = 2C\eta\frac{m}{r}\frac{v^2(t_f)}{\sqrt{\dot{f}_{GW}(t_f)}}e^{i[\psi_f(t_f)-\pi/4]}, \tag{2.4}$$

where t_f is the stationary point [19] and an overdot denotes the derivative w.r.t. t. On substituting for the stationary point t_f, and consistently using the available PN expansions of the flux and energy functions, the phase in Eq. (2.4) takes on the simple form:

$$\psi_f(t_f) = 2\pi f T_C - \phi_C + 2\int_{v_f}^{\infty} dv(v_f^3 - v^3)\frac{dE/dv}{F} = 2\pi f T_C - \phi_C + \sum_0^4 \Psi_k(f)\tau^k, \tag{2.5}$$

where $v_f \equiv v(t_f) = (\pi m f)^{1/3}$, T_C is the (nominal) time of coalescence [20], ϕ_C is the phase at $t = T_C$. The τ^k are the so-called (dimensionless) PN *chirp times* [21, 22]:

$$\tau^0 = \frac{5}{256\eta v_0^5},\ \tau^1 = 0,\ \tau^2 = \frac{5}{192\eta v_0^3}\left(\frac{743}{336} + \frac{11}{4}\eta\right),\ \tau^3 = \frac{\pi}{8\eta v_0^2},\ \tau^4 = \frac{5}{128\eta v_0}\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2\right), \tag{2.6}$$

where $v_0 = (\pi m f_0)^{1/3}$, f_0 is a scaling frequency (to be specified below), and the Ψ_k are given, in terms of the scaled frequency $\nu \equiv f/f_0$, by:

$$\Psi_0 = \frac{6}{5\nu^{5/3}},\ \Psi_1 = 0,\ \Psi_2 = \frac{2}{\nu},\ \Psi_3 = \frac{-3}{\nu^{2/3}},\ \Psi_4 = \frac{6}{\nu^{1/3}}. \tag{2.7}$$

In the *restricted* PN approximation the amplitude in (2.4) depends on frequency simply through a factor $f^{-7/6}$.

III. THE FIRST POST-NEWTONIAN MATCH AND THE STANDARD TEMPLATE BANK

In the stationary phase restricted PN approximation for $\tilde{h}$, the match (or *reduced normalized deflection*, in the jargon of [10]), can be written:

$$\bar{D} = \max_{\Delta T_c} \frac{\left| \int_{f_\mathrm{i}}^{f_\mathrm{s}} df\ \frac{f^{-7/3}}{S_n(f)} e^{i(2\pi f \Delta T_c + \Delta\Psi)} \right|}{\int_{f_\mathrm{i}}^{f_\mathrm{s}} df\ \frac{f^{-7/3}}{S_n(f)}}, \tag{3.1}$$

where $[f_\mathrm{i}, f_\mathrm{s}]$ is the antenna spectral window, ΔT_c is the difference in the coalescence times of the signal and template, and, at the 1PN level of approximation,

$$\Delta\Psi(\nu) = \frac{6}{5\nu^{5/3}}\Delta\tau^0 + \frac{1}{\nu}\Delta\tau^2, \tag{3.2}$$

where $\Delta\tau^i = \tau^i_S - \tau^i_T$, $i = 0, 2$, are the differences in the chirp times of the source (S) and template (T), respectively, $\nu = f/f_0$, and f_0 is the frequency at which $S_n(f)$ is minimum. In the sequel we shall adopt the LIGO form [23] of the (one-sided) noise power spectral density,

$$S_n(\nu) = \frac{S_0}{5}\left[\nu^4 + 2\left(1 + \nu^2\right)\right]. \tag{3.3}$$

The reduced deflection Eq. (3.1) depends on f_i, f_s and f_0 only through the integration limits, which become $\nu_\mathrm{i} \equiv f_\mathrm{i}/f_0$ and $\nu_\mathrm{s} \equiv f_\mathrm{s}/f_0$.

Following Owen [7] we shall perform a coordinate transformation which, asymptotically in a neighbourhood of the peak $\Delta\tau^0 = \Delta\tau^2 = 0$, re-shapes the surface $z = \bar{D}$ into a convex circular paraboloid:

$$\begin{cases} \Delta\tau^0 = \Delta x^0 \cos\vartheta - \Lambda \Delta x^1 \sin\vartheta, \\ \Delta\tau^2 = \Delta x^0 \sin\vartheta + \Lambda \Delta x^1 \cos\vartheta. \end{cases} \tag{3.4}$$

The above transformation depends on the choice of f_0, ν_i and ν_s. In the following, for illustrative purposes, we let $f_0 = 200$ Hz (initial LIGO), $\nu_\mathrm{i} = 0.2$ and $\nu_\mathrm{s} = 4$, whereby $\vartheta \approx 0.489$ and $\Lambda = 15.203$. The resulting function $\bar{D}(\Delta x^0, \Delta x^1)$ is shown in Fig. 1.

In the standard lattice approach, the template spacing δ_L is obtained by enforcing the minimal match condition, whereby for *any* admissible source, there exists *at least one* template such that $\bar{D} \geq \Gamma$. The curves $\bar{D} = \Gamma$, shown in the inset of Fig. 1, are *nearly* circular [24] down to $\Gamma = 0.95$, which includes all values of practical interest for a single-step search. Thus, δ_L is the side-length of the inscribed square. In the inset of Fig. 2 we display δ_L as a function of Γ.

IV. THE CARDINAL INTERPOLATION FOR THE 1-PN MATCH

In the cardinal-interpolation approach the template spacing δ_C is obtained by enforcing the minimal match condition on the cardinal expansion of the match [10]. The $2D$ cardinal expansion of the 1PN match is:

$$\bar{D}_B(\Delta x^0, \Delta x^1) = \sum_{m,n}^{-\infty,+\infty} \bar{D}_B(\Delta x^0_m, \Delta x^1_n)\ \mathrm{sinc}\left[\frac{\pi}{\delta_C}\left(\Delta x^0 - \Delta x^0_m\right)\right] \mathrm{sinc}\left[\frac{\pi}{\delta_C}\left(\Delta x^1 - \Delta x^1_n\right)\right], \tag{4.1}$$

where $\Delta x^0_{m+1} - \Delta x^0_m = \Delta x^1_{n+1} - \Delta x^1_n = \delta_C$ is the cardinal spacing to be determined. This expansion represents *exactly* [10] the function:

$$\bar{D}_B = \mathcal{F}^{-1}_{[\Delta\vec{y}\to\Delta\vec{x}]} \left\{ W\left(\frac{\Delta\vec{y}}{B}\right) \mathcal{F}_{[\Delta\vec{x}\to\Delta\vec{y}]} \bar{D}(\Delta\vec{x}) \right\}, \tag{4.2}$$

where $\mathcal{F}$ is the Fourier-transform operator,

$$W\left(\frac{\Delta y^0}{B}, \frac{\Delta y^1}{B}\right) = \begin{cases} 1, & |\Delta y^0| < B, |\Delta y^1| < B, \\ 0, & \text{elsewhere,} \end{cases} \tag{4.3}$$

and $B = (2\delta_C)^{-1}$. Similar to the Newtonian case [10], the 1PN match Eq. (3.1) is a quasi-band-limited function in the L^∞ norm, namely,

$$\exists \gamma, B_c \in \mathcal{R}^+ : \sup|\bar{D} - \bar{D}_B| = \exp[-\gamma(B - B_c)]. \tag{4.4}$$

The cardinal expansion Eq. (4.1) can be shown to *approximate* the function Eq. (3.1) in the L^∞ (and L^2) norm, using the *minimum* 2D sample density $1/\delta_C^2$ compatible with a prescribed accuracy (see [10], and references cited therein). The *exponential* decay of the error (4.4) is exemplified in Fig. 3.

We shall now discuss the template spacing to be used in Eq. (4.1), for a prescribed minimal match Γ.

The source and template parameters can be always conveniently written as follows:

$$\vec{x}_S = \vec{x}_Q + \vec{\alpha}_S \delta_C, \quad \vec{x}_T = \vec{x}_Q + \vec{\alpha}_T \delta_C, \tag{4.5}$$

where $\alpha^{0,1}_{S,T} \in [0, 1]$ and $\vec{x}_Q$ correspond to the lower left corner of a suitable template-space cell.

For any given value of δ_C, as the source coordinates $\vec{x}_S$ move in the template-space cell, the location $\vec{x}^{\max}_T$ where the cardinal-interpolated match $\bar{D}_B$ attains its maximum value and the maximum value itself $\bar{D}^{\max}_B$ change. The minimal match condition should be enforced for the special value $\vec{\alpha}_*$ of $\vec{\alpha}_S$ where the interpolated *maximum* is a *minimum*.

Numerical experiments show that $\vec{\alpha}_* = (0.5, 0.5)$, which corresponds to the center of the cell, below some critical value $\delta_\times$ of δ_C. Above this critical value, a bifurcation occurs, i.e., *two* (equal) minima exist [25] located on the descending diagonal of the cell, where

$$\alpha^0_S = 0.5 + \xi, \quad \alpha^1_S = 0.5 - \xi, \tag{4.6}$$

symmetrically with respect to the cell center, viz. at:

$$\xi = \pm\xi_*(\delta_C), \qquad \xi_*(\delta_C) \in [0, 0.5]. \tag{4.7}$$

This is illustrated in Fig. 4, where we show the density plots of $\bar{D}^{\max}_B$ as a function of $\vec{\alpha}_S$ in the fundamental cell, for the representative cases $\delta_C = 0.078 < \delta_\times$ and $\delta_C = 0.099 > \delta_\times$.

The values of $\bar{D}^{\max}_B$ on the descending diagonal of the cell, where the minima are located and Eq. (4.6) holds, are plotted in Fig. 5, as a function of the displacement ξ off the cell center, for several values of δ_C. The values of $\bar{D}^{\max}_B$ at the minima of these curves, occurring at $\xi = \xi_*(\delta_C)$, are the minimal matches.

It is seen, e.g., that for $\Gamma = 0.97$ one has $\delta_C \approx 0.104$. By comparison with Fig. 2, where $\delta_L = 0.052$ for $\Gamma = 0.97$, we deduce that $\delta_C/\delta_L \approx 2$. Hence the reduction in the 2D template density is of the order of $(\delta_C/\delta_L)^2 \sim 4$, which is quite remarkable. It is also seen that, similar to the 0PN case, the 1PN template density reduction with respect to the plain lattice increases with Γ, as shown in Fig. 2.

V. CONCLUSIONS

The number of 1PN templates required to keep the minimal match above a given threshold Γ can be significantly (by a factor ≈ 4 at $\Gamma \approx 0.97$) reduced by using cardinal interpolation. The statistical properties of the 1PN cardinal-interpolated (partially maximised) correlator bank will be the subject of a forthcoming paper. Extension to 2.5PN templates should be straightforward, in principle, using the (almost-flat parameter space manifold) coordinates introduced by Tagoshi and Tanaka [26].

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Figure Captions

Fig. 1 - The function $z = \bar{D}(\Delta x^0, \Delta x^2)$, and some of its contour levels (inset).

Fig. 2 - The plain 1PN lattice spacing δ_L (inset), and the $2D$ template density reduction $(\delta_C/\delta_S)^2$, achieved by use of cardinal interpolation, both plotted as functions of the match Γ.

Fig. 3 - The L^∞ error as a function of δ_C^{-2}.

Fig. 4 - Density plot of $\bar{D}^{\max}$ vs. $\vec{\alpha}_S$ for $\delta_C = 0.0078$ (left) and $\delta_C = 0.0099$ (right).

Fig. 5 - $\bar{D}_B^{\max}$ vs. ξ, for several values of δ_C.

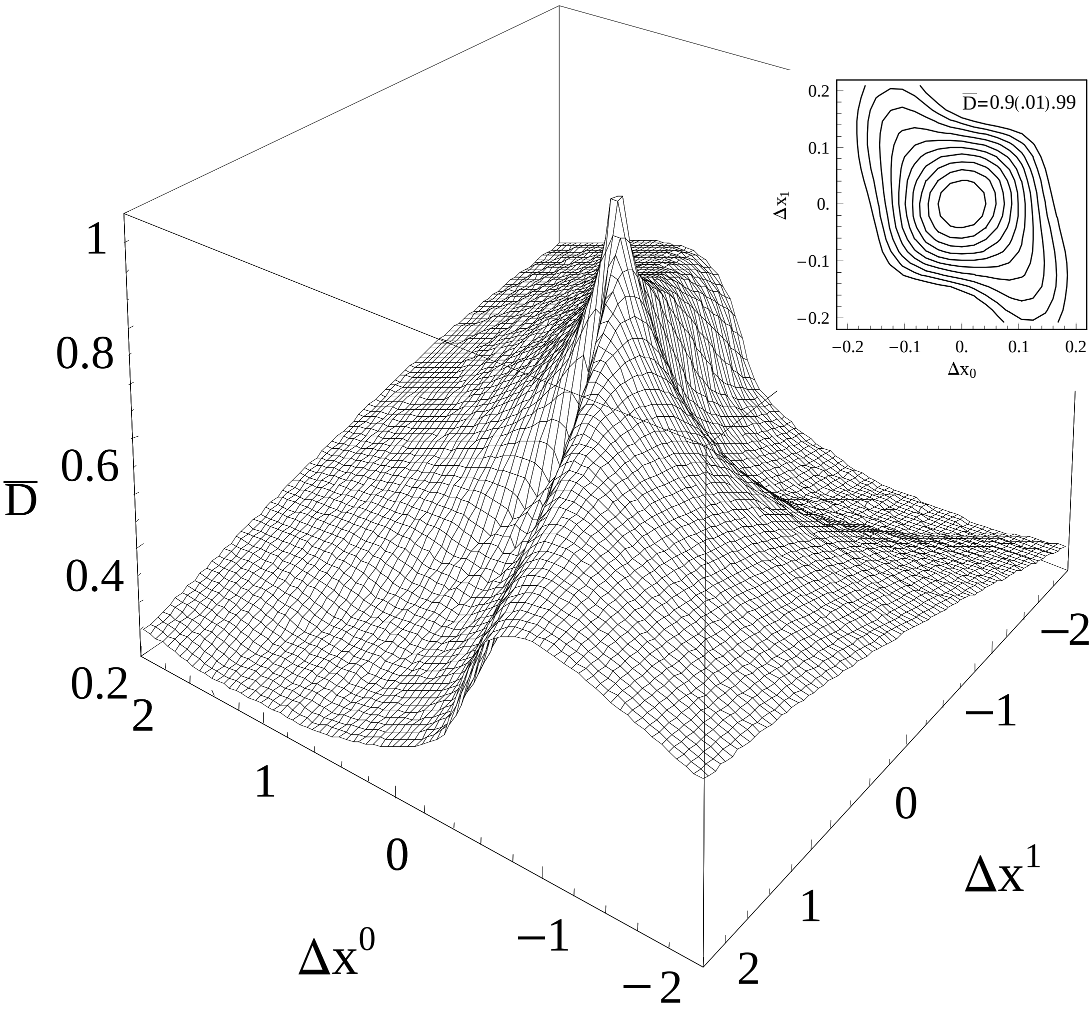

Fig. 1 - The function $z = \overline{D}(\Delta x^0, \Delta x^1)$,
and some of its contour levels (inset).

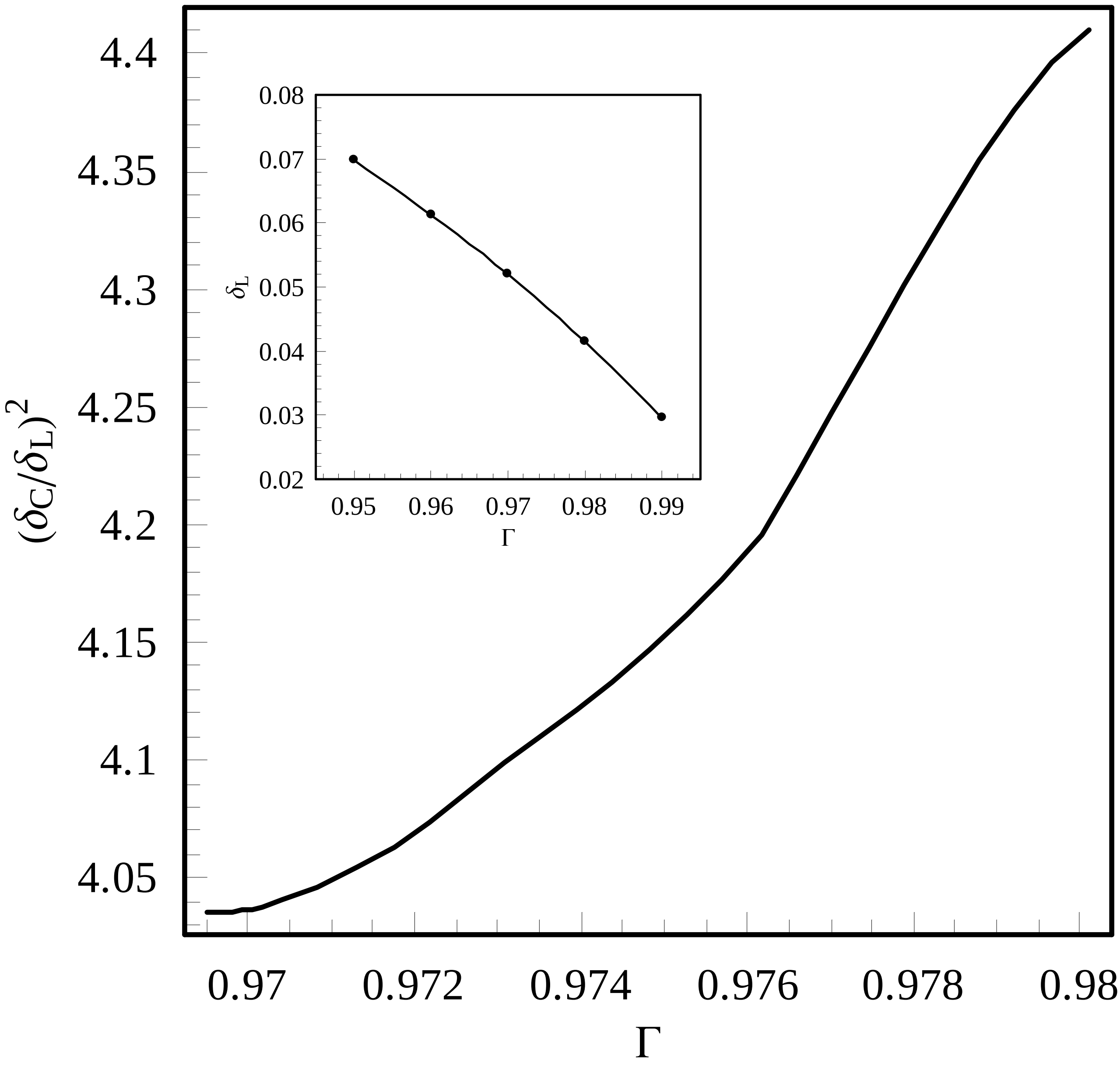

Fig. 2 – The plain 1PN lattice spacing δ_L (inset) and the 2D template density reduction $(\delta_L/\delta_L)^2$ achieved by use of cardinal interpolation, both plotted as functions of the minimal match Γ.

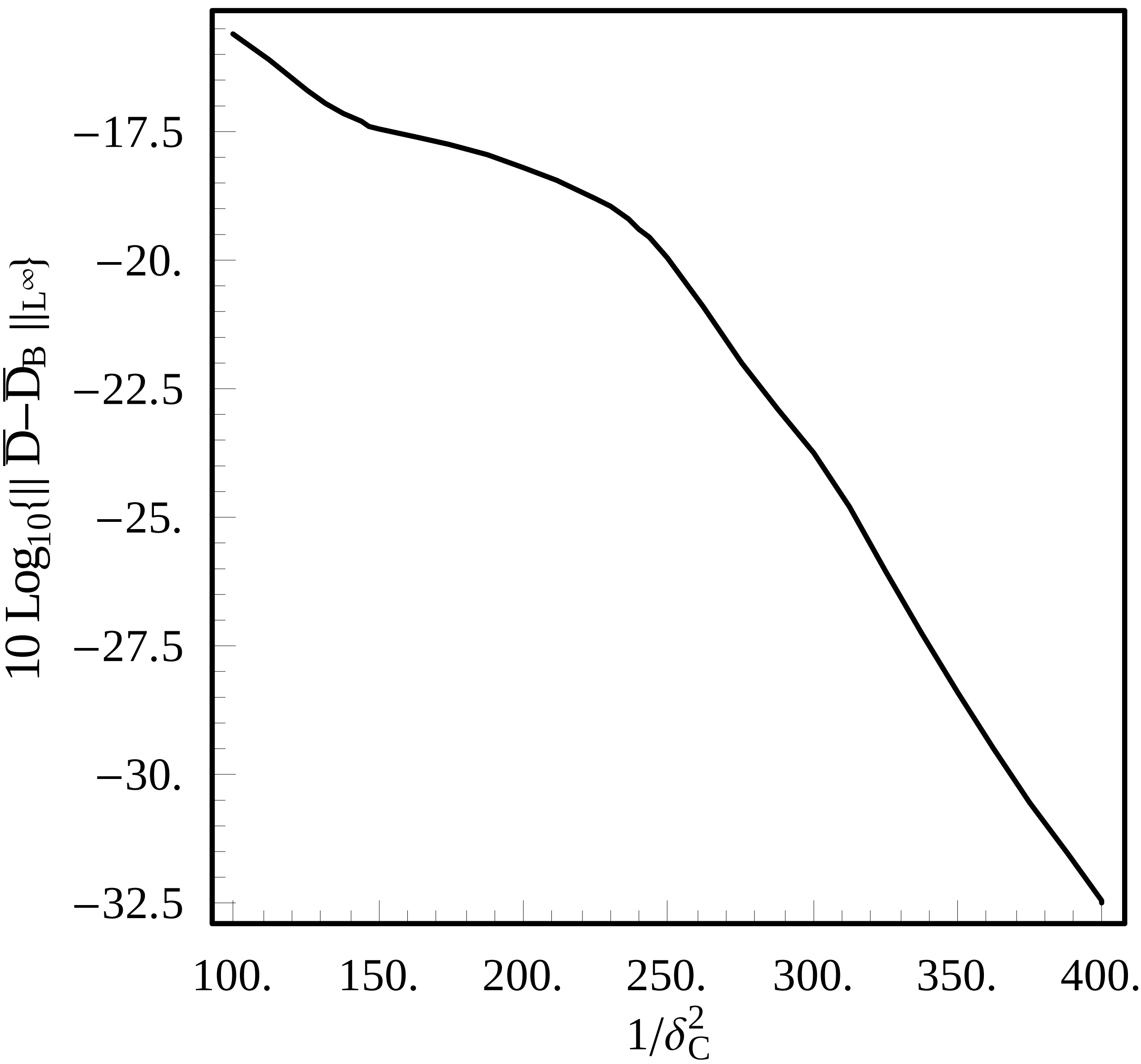

Fig. 3 – The L^{∞} error as a function of δ_C^{-1}.

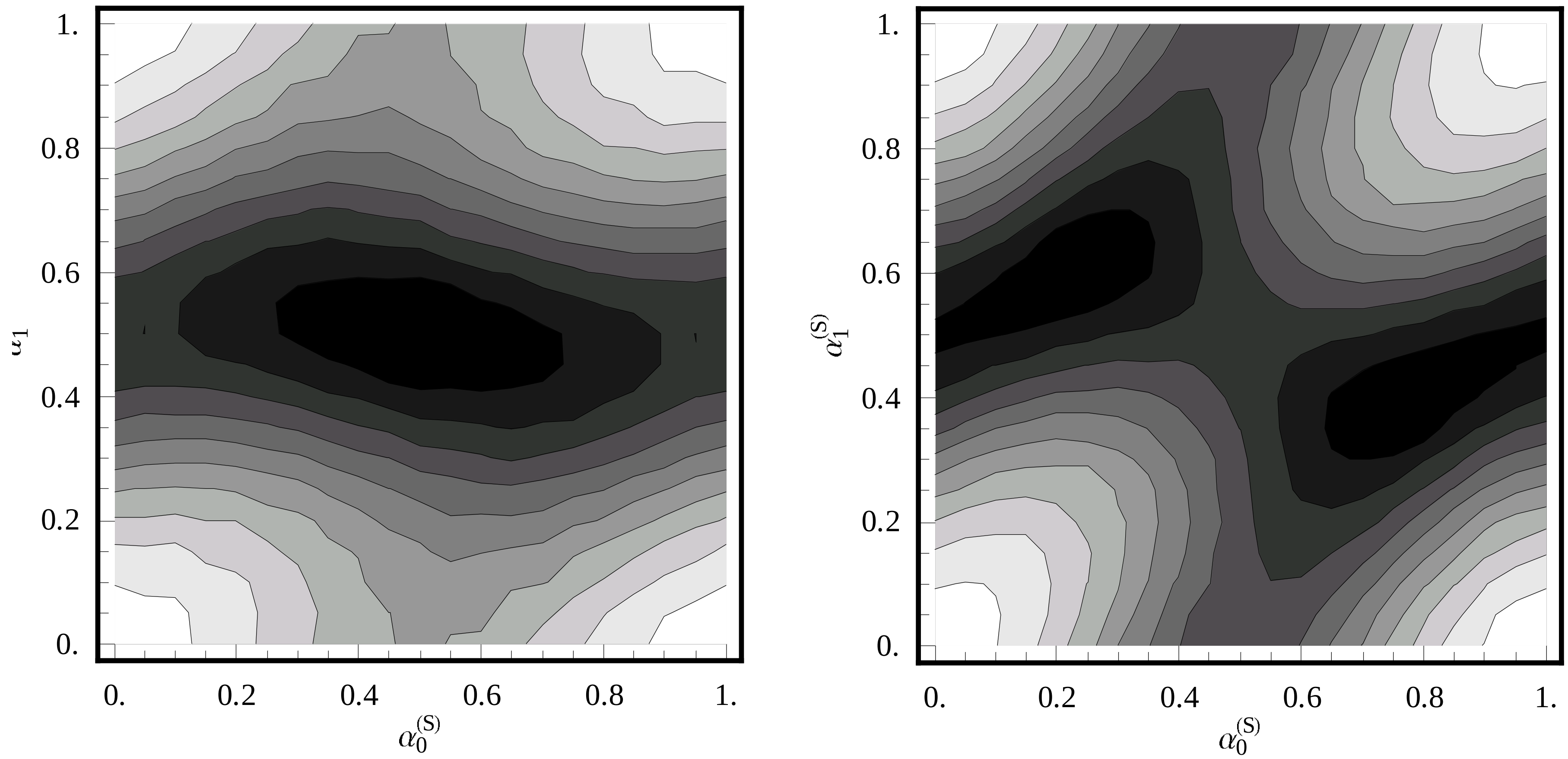

Fig. 4 - Density plots of $\overline{D}_B^{max}$ vs. $\alpha^{(S)}$. at δ_C =0.0078 (left) and δ_C =0.0099 (right).

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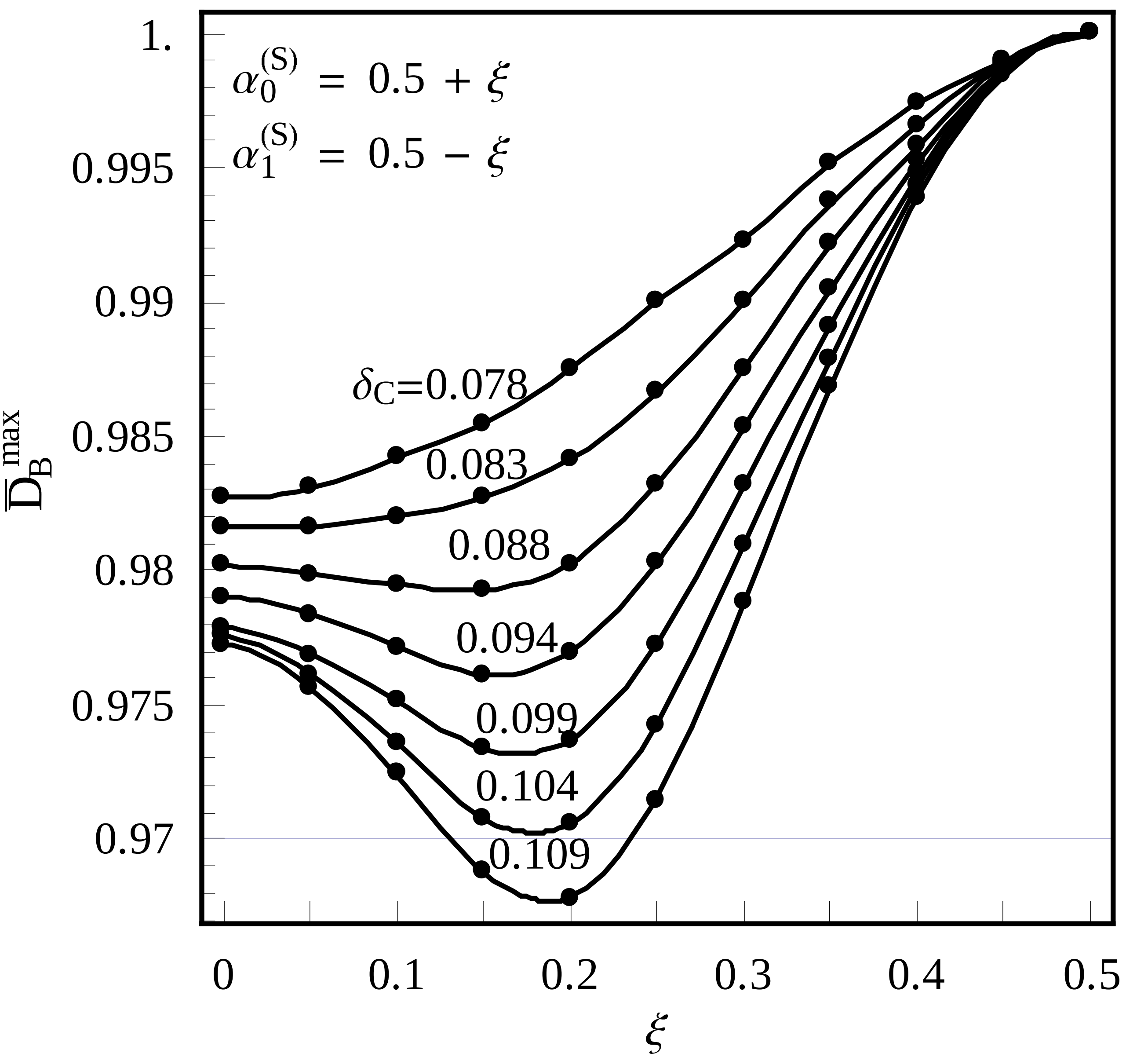

Fig. 5 - $\overline{D}_B^{max}$ vs. ξ for several values of δ_C.